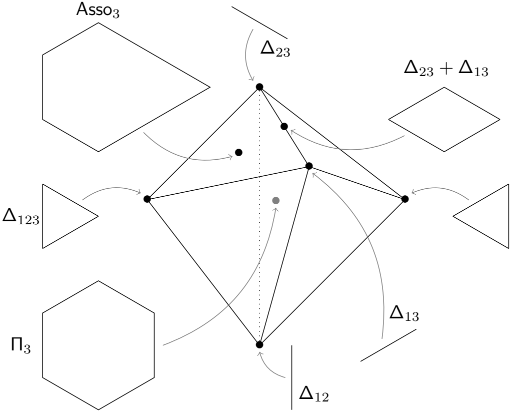
Deformed permutahedra
An inductive construction of the submodular cone

Germain Poullot with Georg Loho & Arnau Padrol



1 October 2025, Stockholm

1. Deformations (a.k.a. Minkowski summands)
   - Deformations
   - Cone of deformations
2. Deformed permutahedra = submodularity
   - Combinatorics of the permutahedron
   - Submodular functions
3. Submodular Cone in general
   - Known facts about $\mathbb{SC}_n$
   - Submodular cone $n = 4$ (and $n = 5$)
4. Inductive construction of (rays of) $\mathbb{SC}_n$
   - GP-sum
   - Rays of $\mathbb{SC}_n$

Deformations (a.k.a. Minkowski summands)

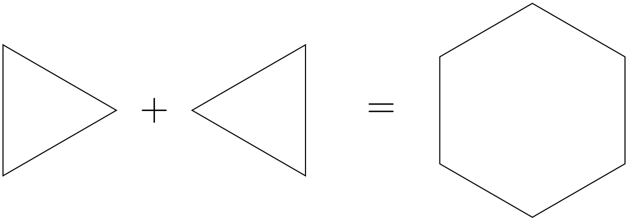
Minkowski sum

Definition

P, Q polytopes. *Minkowski sum*:

$$\mathrm{P} + \mathrm{Q} = \{\boldsymbol{p} + \boldsymbol{q} \;\; ; \;\; \boldsymbol{p} \in \mathrm{P}, \;\; \boldsymbol{q} \in \mathrm{Q}\}$$

N.B. $\mathrm{Vert}(\mathrm{P} + \mathrm{Q}) \subseteq \mathrm{Vert}(\mathrm{P}) + \mathrm{Vert}(\mathrm{Q})$



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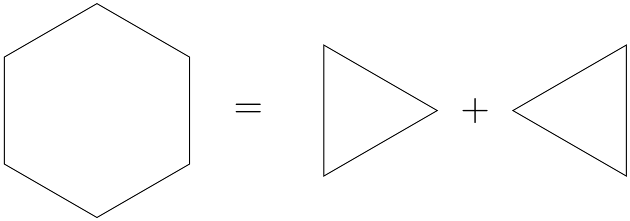

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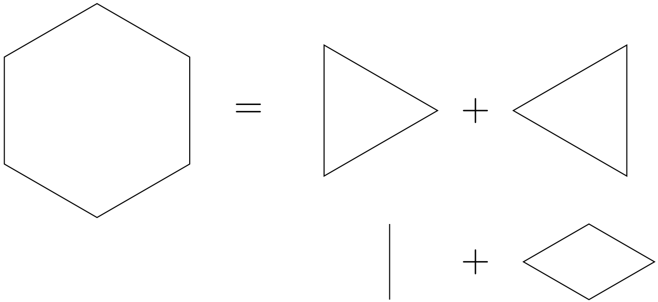

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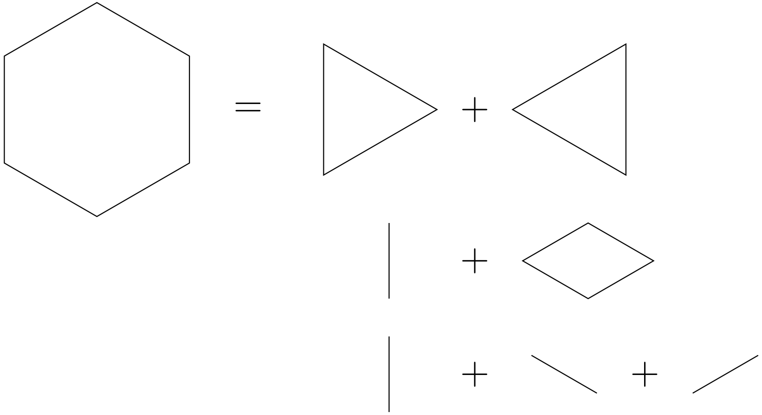

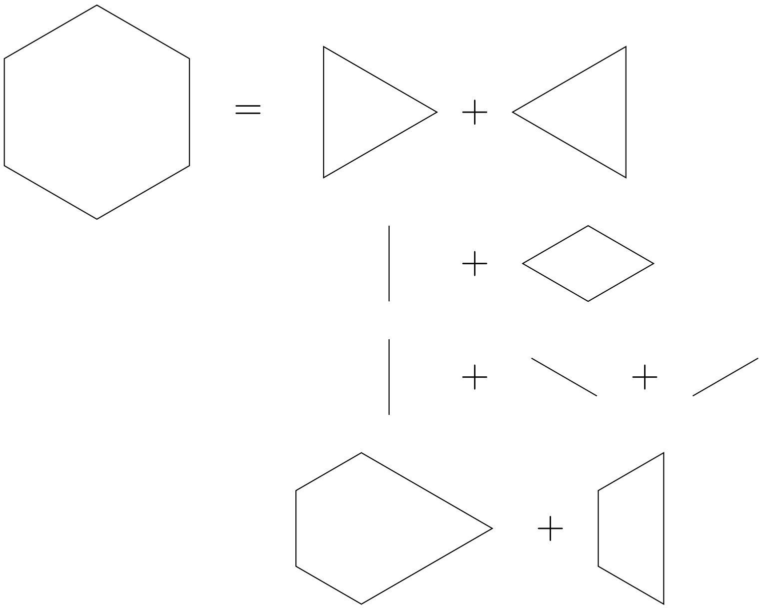

Minkowski summands

Definition

Q is a *Minkowski summand*, a.k.a. *deformation*, of P when there exists R and $\lambda > 0$ with:

$$\lambda \mathsf{P} = \mathsf{Q} + \mathsf{R}$$

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What is the best way to write P as a Minkowski sum ?

- With the fewest number of (indecomposable) summands ?
- With the (indecomposable) summands of smallest dimension ?
- Respecting some symmetries ?
- ...

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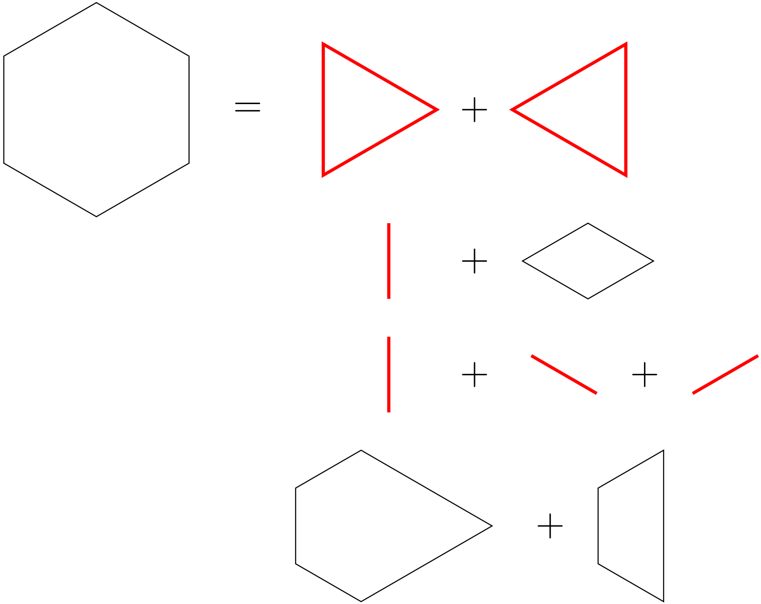
$\Longrightarrow$ What is the structure of $\mathbb{DC}(\mathrm{P})$?

Minkowski summands

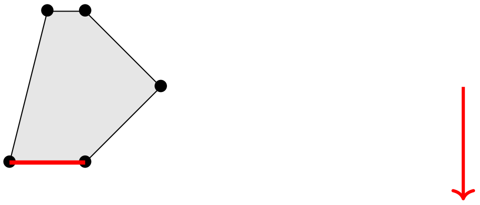


Definition

Polytope: convex hull of finitely many points in $\mathbb{R}^n$
bounded intersection of finitly many half-spaces in $\mathbb{R}^n$

Definition

Face: $\mathrm{P}^{\boldsymbol{c}} := \left\{ \boldsymbol{x} \in \mathbb{R}^n \ ; \ \langle \boldsymbol{x}, \boldsymbol{c} \rangle = \max_{\boldsymbol{y} \in \mathrm{P}} \langle \boldsymbol{y}, \boldsymbol{c} \rangle \right\}$



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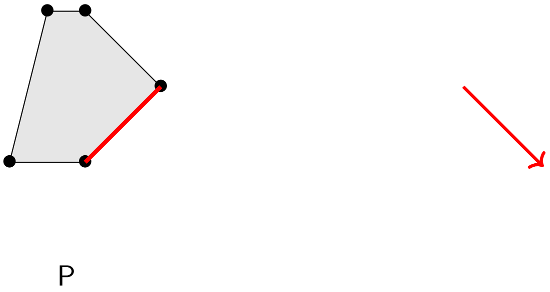

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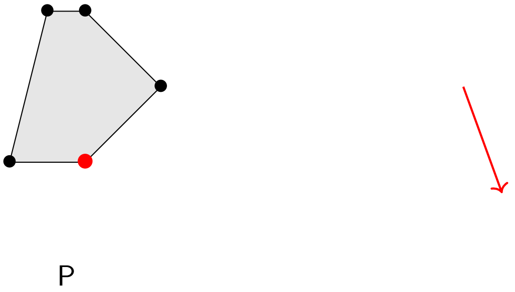
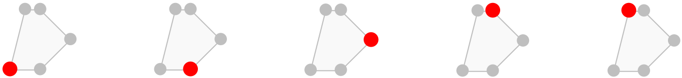


Face lattice: poset of inclusions of faces



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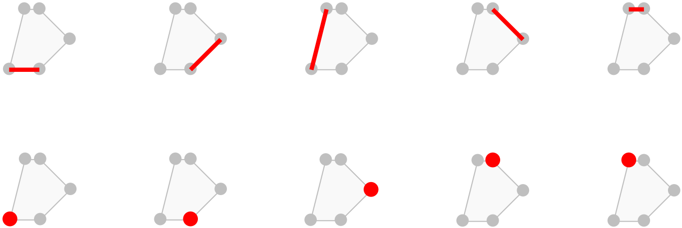

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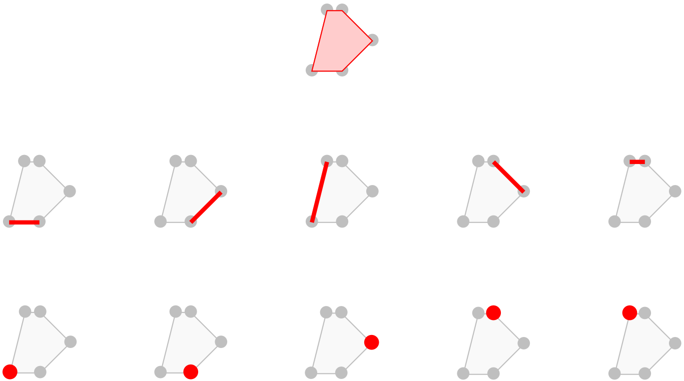

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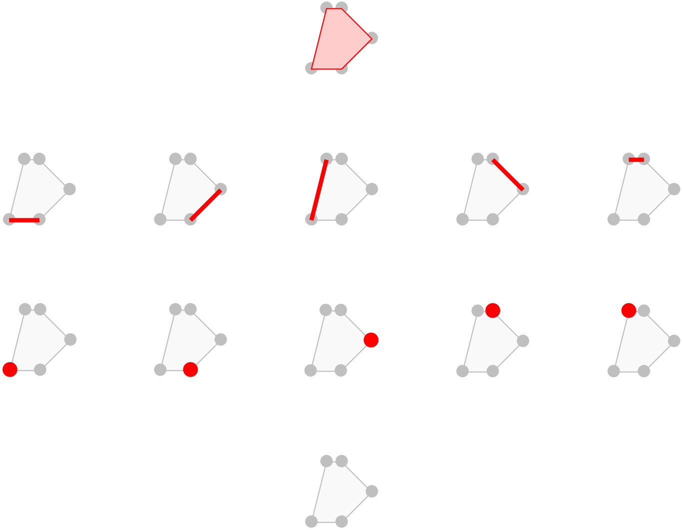

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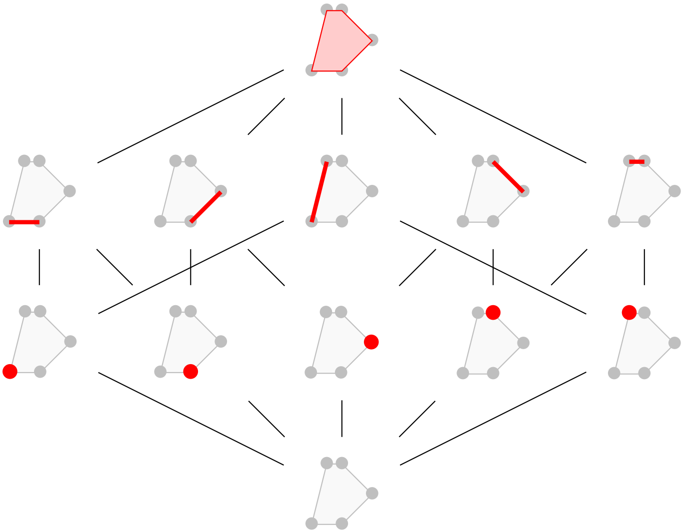

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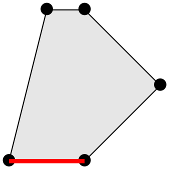
Normal cone of a face F: $\mathcal{N}_{\mathrm{P}}(\mathrm{F}) := \{\boldsymbol{c} \; ; \; \mathrm{P}^{\boldsymbol{c}} = \mathrm{F}\}$
Normal fan $\mathcal{N}_{\mathrm{P}}$: collection of $\mathcal{N}_{\mathrm{P}}(\mathrm{F})$ for F face of P



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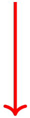

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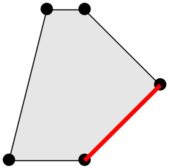

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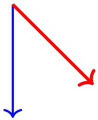

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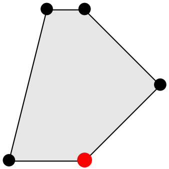

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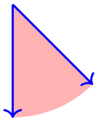

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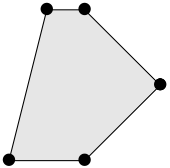

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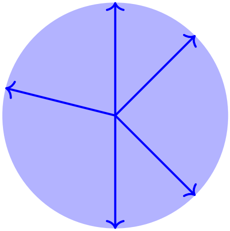
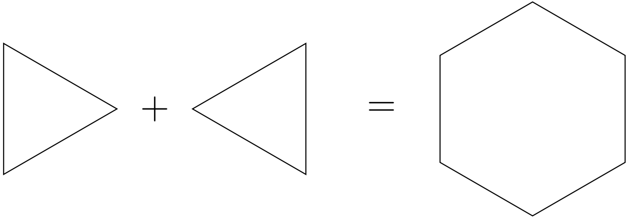


$\mathcal{N}_P$

Lemma

$\mathcal{N}_{\mathsf{P+Q}} =$ common refinement of $\mathcal{N}_{\mathsf{P}}$ and $\mathcal{N}_{\mathsf{Q}}$



Minkowski sum seen with normal fans

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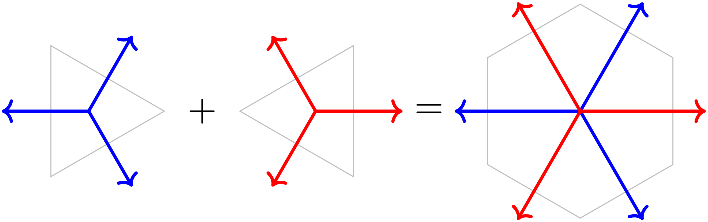
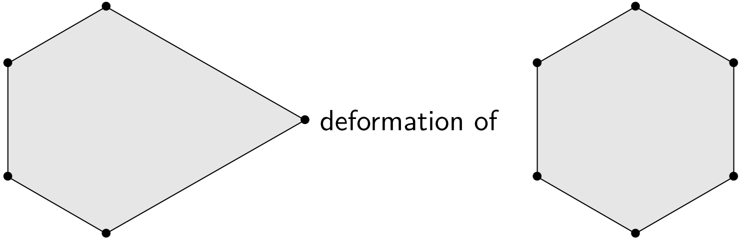


$\mathcal{N}_{\mathsf{P}}$: Normal fan of polytope P

Coarsening: Choose maximal cones and merge them

Theorem

Q *is a deformation of* P *iff* $\mathcal{N}_{\mathsf{Q}}$ *coarsens* $\mathcal{N}_{\mathsf{P}}$.



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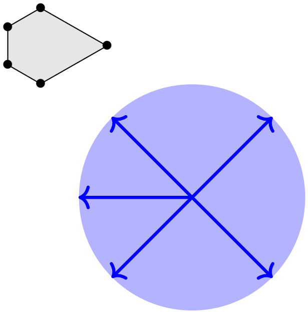

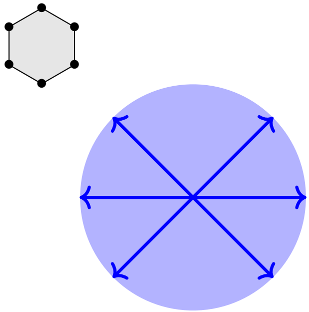

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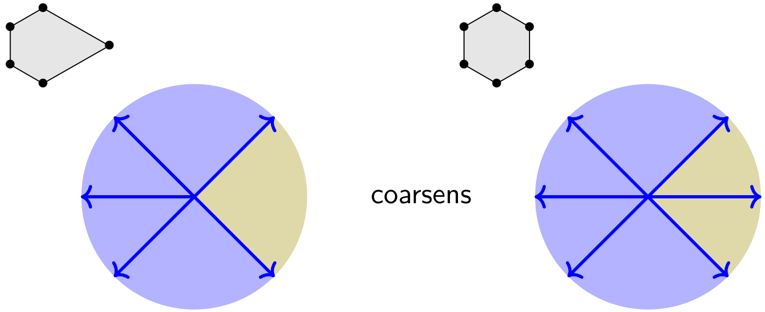
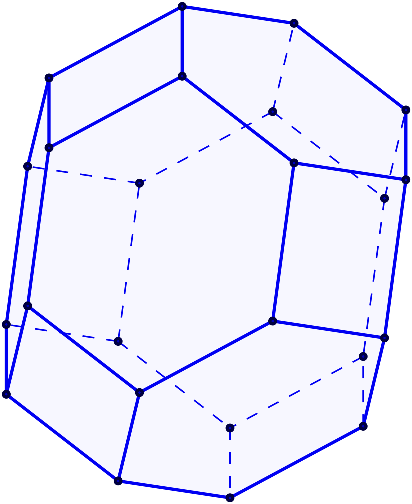
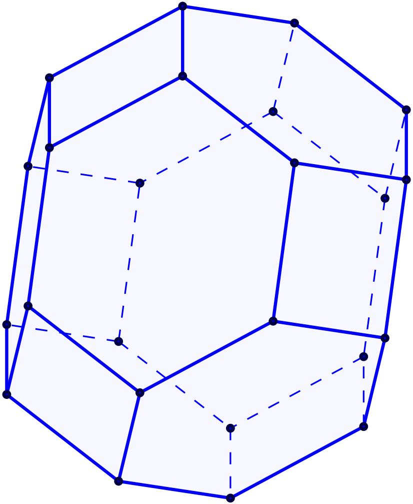




Permutahedron Π_4



Sequence of deformations of Π_4

Height deformation cone

Theorem

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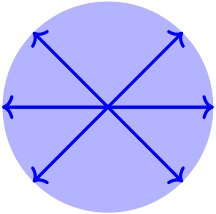

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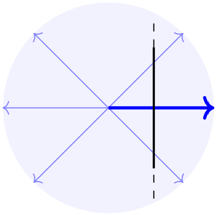

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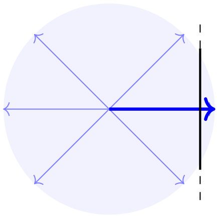

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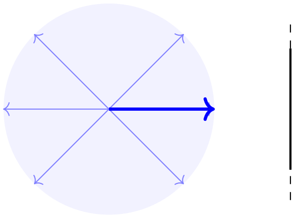

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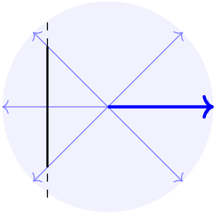

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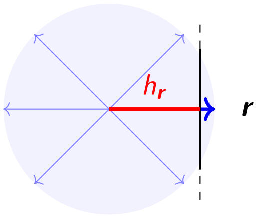

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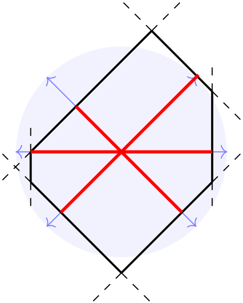

Parametrization:

height vector:
$\boldsymbol{h} = (h_{\boldsymbol{r}})_{\boldsymbol{r} \text{ rays}}$

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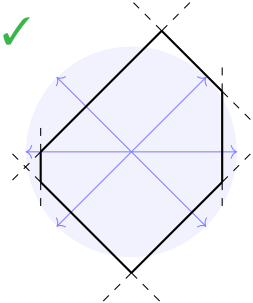

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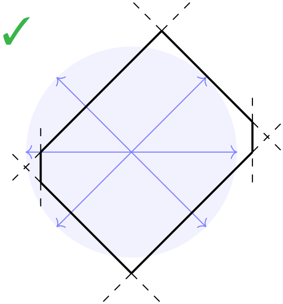

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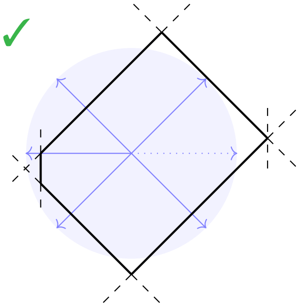

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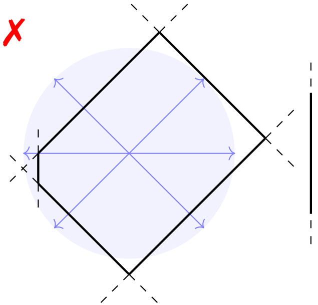

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Wall-crossing inequalities:

linear inequalities on $\boldsymbol{h}$

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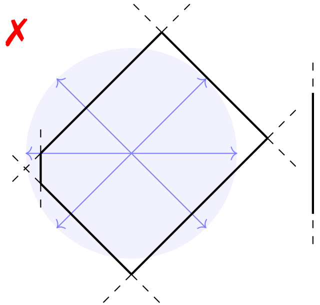

Parametrization:

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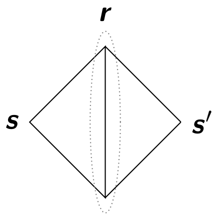
Wall-crossing inequalities:
linear inequalities on $\boldsymbol{h}$

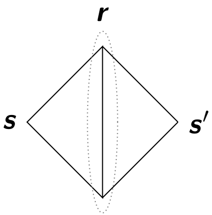
$$P_{\boldsymbol{h}} = \{\boldsymbol{x}\ ;\ \langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle \leq h_{\boldsymbol{r}}\}$$

Wall-crossing inequalities



If, with $\alpha_{\boldsymbol{s}}, \alpha_{\boldsymbol{s}'} > 0$

$$\alpha_{\boldsymbol{s}} \boldsymbol{s} + \alpha_{\boldsymbol{s}'} \boldsymbol{s}' + \sum_{\boldsymbol{r}} \alpha_{\boldsymbol{r}} \boldsymbol{r} = \boldsymbol{0}$$



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then

$$\alpha_{\boldsymbol{s}} h_{\boldsymbol{s}} + \alpha_{\boldsymbol{s}'} h_{\boldsymbol{s}'} + \sum_{\boldsymbol{r}} \alpha_{\boldsymbol{r}} h_{\boldsymbol{r}} \geq 0$$

Summary on $\mathbb{DC}(\mathsf{P})$

$\mathbb{DC}(\mathsf{P})$

Q Minkowski summands	ℓ edge-lengths	$\boldsymbol{h}$ heights on rays
$\mathsf{Q}_1 + \mathsf{Q}_2$ Dilation $\lambda \mathsf{Q}$ Translations	$\ell_1 + \ell_2$ $\lambda \ell$ None	$\boldsymbol{h}_1 + \boldsymbol{h}_2$ $\lambda \boldsymbol{h}$ Lineal
complicated	edge directions Polygonal face eq. V-description	normal fan $\mathcal{N}_\mathsf{P}$ Wall-crossing ineq. H-description
Polytope algebra	Weight algebra	Polynomial algebra

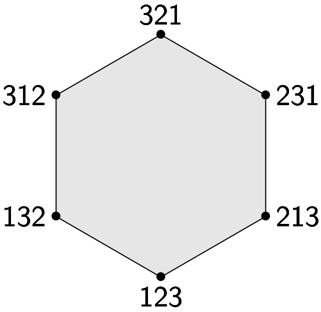
$\mathbb{DC}(\mathsf{P})$ is a ray = P Minkowski indecomposable

$\mathbb{DC}(\mathsf{P})$ is simplicial cone = P has **unique** Minkowski decomposition

Deformed permutahedra = submodularity

Example (Permutahedron)

$$\Pi_n = \operatorname{conv}\left\{ \begin{pmatrix} \sigma(1) \\ \vdots \\ \sigma(n) \end{pmatrix} \; ; \; \sigma \text{ permutation of } \{1, \dots, n\} \right\}$$



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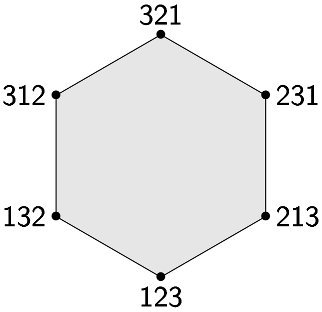

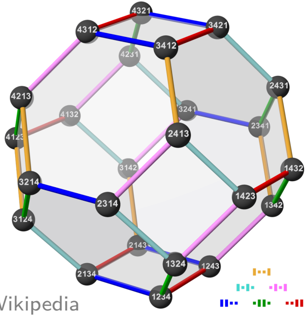

Wikipedia

Braid fan

Definition

Braid fan: arrangement of hyperplanes $H_{i,j} := \{\boldsymbol{x}\ ;\ x_i = x_j\}$

Braid fan

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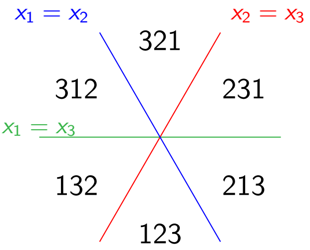
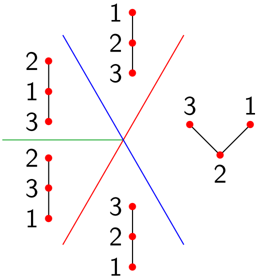
Definition

Generalized permutahedron: deformation of Π_n

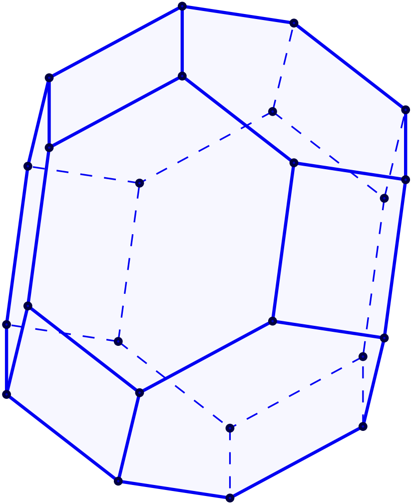
i.e. P generalized permutahedron iff $\mathcal{N}_{\mathrm{P}}$ coarsens braid fan

i.e. P generalized permutatahedron iff edges in directions $\boldsymbol{e}_i - \boldsymbol{e}_j$

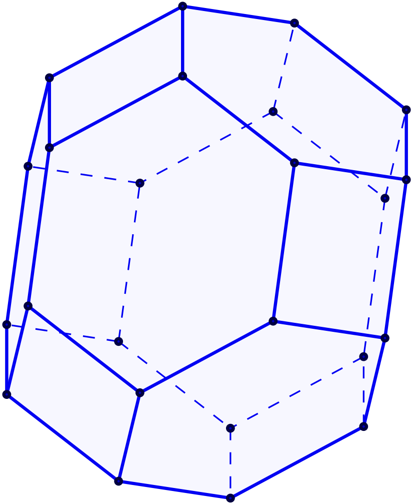




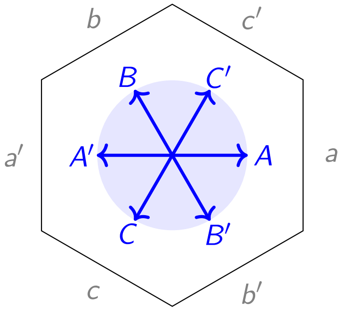
Deformations of Π_4



Permutahedron Π_4



Sequence of deformations of Π_4



Wall-crossing inequalities:

$h_A + h_B \geq h_{C'}$
$h_B + h_C \geq h_{A'}$
$h_C + h_A \geq h_{B'}$
& 3 others ineq.

Polygonal face equations:

$\ell_a - \ell_{a'} = \ell_b - \ell_{b'} = \ell_c - \ell_{c'}$
& $\boldsymbol{\ell} \in \mathbb{R}^6_+$

Submodular functions

Definition

Submodular functions $\boldsymbol{h} : 2^{[n]} \to \mathbb{R}$

$$\forall A, B \subseteq [n],\ \ h(A) + h(B) \geq h(A \cap B) + h(A \cup B)$$

In bijection with generalized permutahedra:
$\mathrm{P}_{\boldsymbol{h}} = \{\boldsymbol{x}\ ;\ \ \sum_{i \in A} x_i \leq h(A) \text{ for all } A \subseteq [n]\}$

Submodular Cone in general

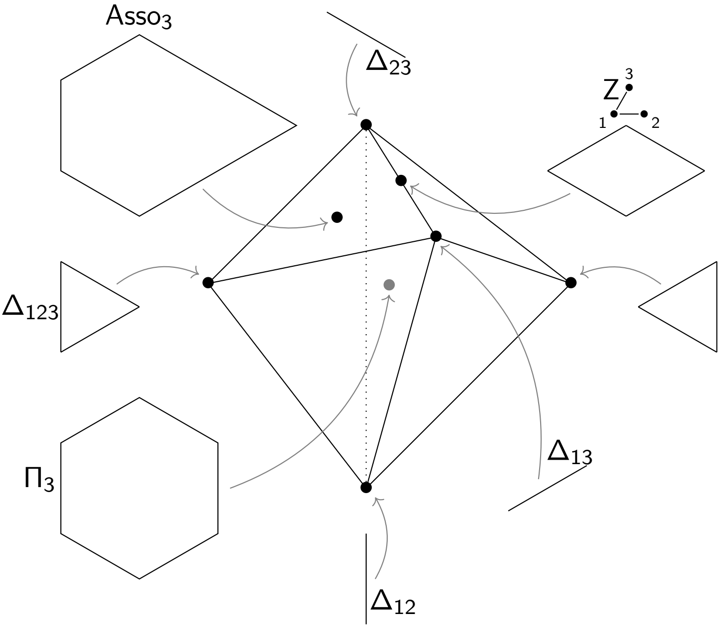
Submodular Cone

Definition

Submodular cone: deformation cone of the permutahedron Π_n

	$\mathbb{SC}_n$
Dim (no lineal)	$2^n - n - 1$
# facets	$\binom{n}{2}2^{n-2}$
# rays	unknown!

Submodular Cone for Π_3



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	$\mathbb{SC}_n$
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But that belongs to a different presentation...

Submodular Cone's faces

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Theorem (Faces of $\mathbb{DC}(\mathrm{P})$)

If Q *deformation of* P, *then* $\mathbb{DC}(\mathrm{Q})$ *is a face of* $\mathbb{DC}(\mathrm{P})$.

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	$\mathbb{SC}_n$	$\mathbb{DC}(\mathrm{Asso}_n)$
Dim (no lineal)	$2^n - n - 1$	$\binom{n}{2}$
# facets	$\binom{n}{2}2^{n-2}$	$\binom{n}{2}$
# rays	unknown!	$\binom{n}{2}$
		simplicial!

But that belongs to a different presentation...

Submodular Cone's faces

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	$\mathbb{SC}_n$	$\mathbb{DC}(\mathrm{Asso}_n)$	Graph zono.	Nestohedra
Dim (no lineal)	$2^n - n - 1$	$\binom{n}{2}$	✓	✓
# facets	$\binom{n}{2}2^{n-2}$	$\binom{n}{2}$	✓	✓
# rays	unknown!	$\binom{n}{2}$	✗	✗
		simplicial!		

But that belongs to a different presentation...

Recall: dim $= 11$, nbr facets $= 80$

Drawing $\mathbb{SC}_4$

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Draw all generalized permutahedra ? (ask computer)

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22 107 faces (Please do not draw...)

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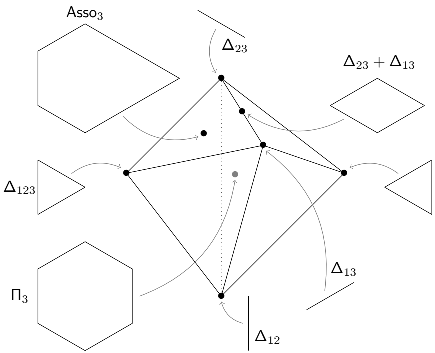
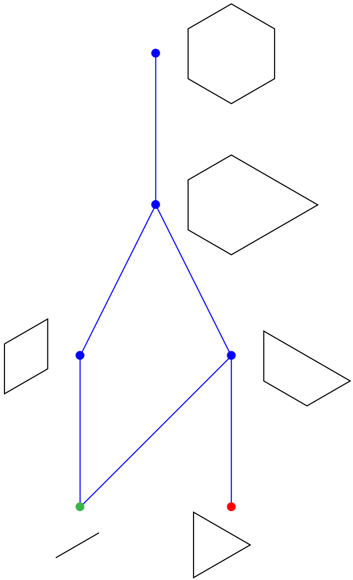
22 107 faces (Please do not draw...)

$\Longrightarrow$ quotient by symmetries

Braid symmetries: permutation of coordinates + central symmetry





Recall: $\dim = 11$, nbr facets $= 80$

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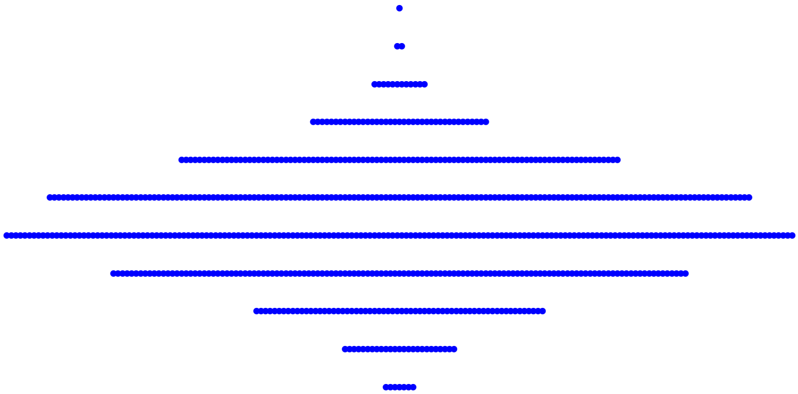
Draw all generalized permutahedra ? (ask computer)

$22\,107$ faces

$\implies$ quotient by symmetries

703 "faces"

Reduced face lattice of $\mathbb{SC}_4$



d	1	2	3	4	5	6	7	8	9	10	11	total
f_d	7	25	64	127	174	155	97	39	12	2	1	703

Reduced f-vector of $\mathbb{SC}_n$

Reduced $\mathbb{SC}_n$ f-vector:

$n = 3$
$\dim \mathbb{SC}_3 = 4$
$(2, 2, 1, 1)$

$n = 4$
$\dim \mathbb{SC}_4 = 11$
$(7, 25, 64, 127, 174,$
$155, 97, 39, 12, 2, 1)$

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$n = 4$
dim $\mathbb{SC}_4 = 11$
$(7, 25, 64, 127, 174,$
$155, 97, 39, 12, 2, 1)$

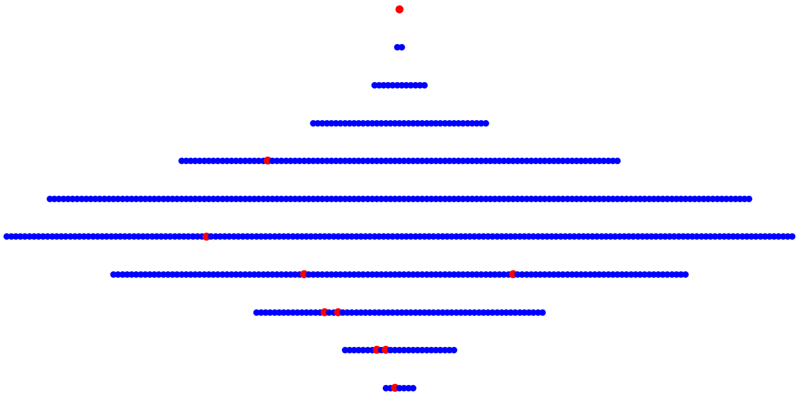
Thanks to Winfried Bruns for helping compute!

Database for dim 1-4 & 19-26

$n = 5$, dim $\mathbb{SC}_5 = 26$

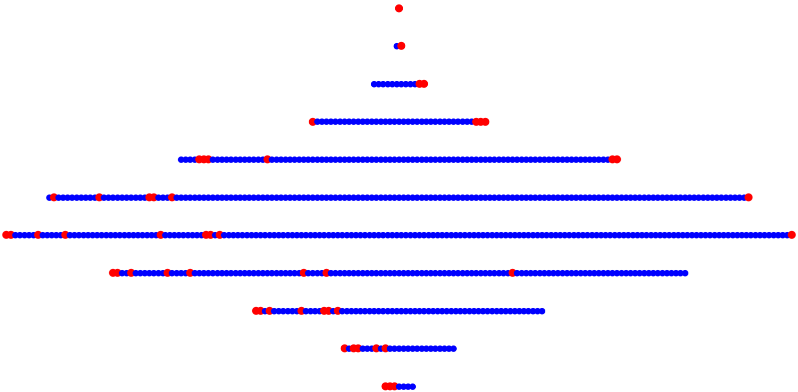
∗672
∗24 026
∗373 433
∗3 355 348
19 739 627
81 728 494
249 483 675
579 755 845
1 048 953 035
1 501 555 944
1 719 688 853
1 587 510 812
1 186 372 740
719 012 097
353 190 577
140 265 886
44 831 594
11 464 559
∗2 326 596
∗372 031
∗46 330
∗4 572
∗355
∗30
∗2
∗1

Graphical zonotopes & Nestohedra are sparse



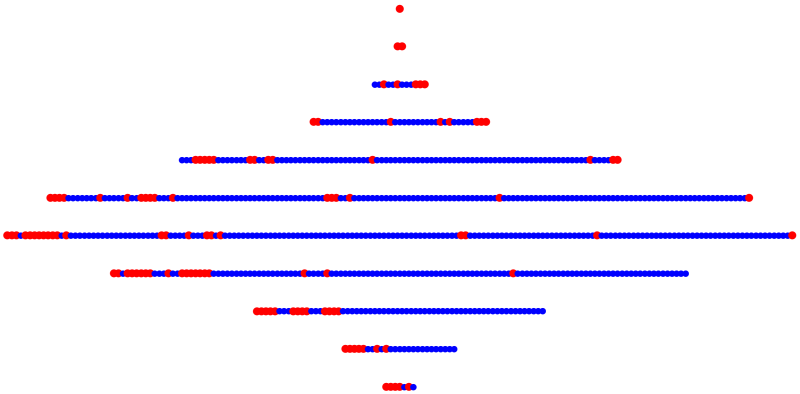
With: Graphical Zonotopes
10 polytopes

Graphical zonotopes & Nestohedra are sparse



With: Graphical Zonotopes & Nestohedra
10 + 46 polytopes

Everything is quite negligible...



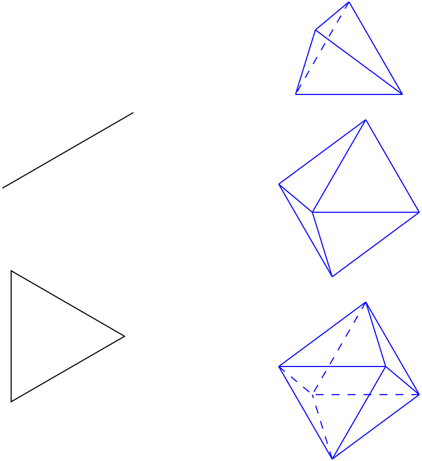
With: Graphical Zono & Nestohedra $\subsetneq$ Hypergraphic Polytopes,

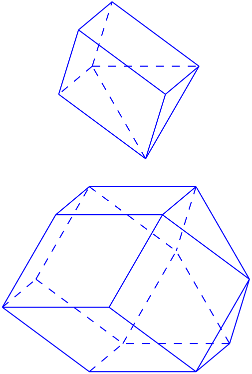
$+$ Shard Polytopes, Quotientopes,

$+$ Matroid Polytopes

$=$ 112 polytope (only...)

What about the rays of $\mathbb{SC}_4$?



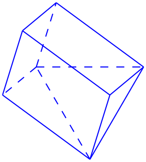


Then 665 of dim 4,
$>$ 126 629 of dim 6

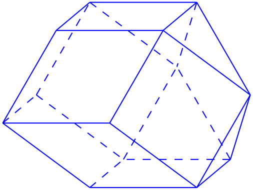
Deformed permutahedra:

Minkowski indecomposable ✔

Matroid Polytopes ✗



Strawberry



Persimmon

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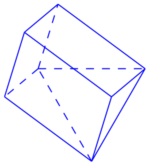

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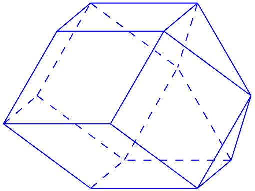

Persimmon

Deformed permutahedra:

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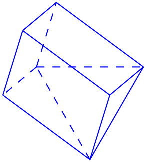
Matroid Polytopes ✗
Hypergraphic polytopes ✗
Shard Polytopes ✗

Persimmon:
Polypositroid ✗
Removahedron ✗



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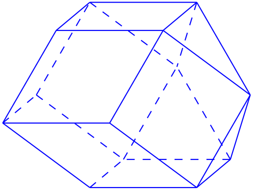

Persimmon

Number of rays of $\mathbb{SC}_n$

Theorem ((approximately) Nguyen, '78)

The number t_n of rays of $\mathbb{SC}_n$ satisfies

$$2^{2^n \cdot n^{-3/2}} \leq t_n$$

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Theorem (Loho–Padrol–P., '25+)

The number t_n of rays of $\mathbb{SC}_n$ satisfies

$$2^{2^n} \leq t_n$$

N.B.: By Upper Bound Theorem: $t_n \leq n^{2^n}$.

Precisely: $n - 2 \leq \log_2 \log_2 t_n \leq n + \log_2 \log_2 n + 1$

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Proof:

What's 2^{2^n}?

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Let $a_n = 2^{2^n}$.

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$\Rightarrow$ I want to construct 1 rays of $\mathbb{SC}_{n+1}$ for each **pair** of rays of $\mathbb{SC}_n$

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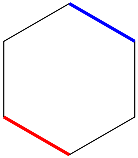
Let's do an induction!

Inductive construction of (rays of) $\mathbb{SC}_n$

GP-sum

P^+, P^-: **opposite** faces, maximizing/minimizing $\boldsymbol{e}_{n+1}$ on P.



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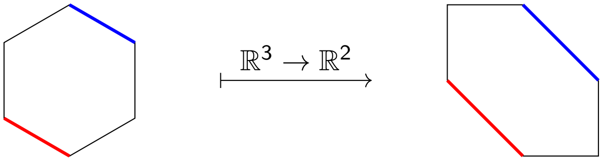
Technical detail, projection $\mathbb{R}^{n+1} \to \mathbb{R}^n$:



${\color{blue}P^+}$, ${\color{red}P^-}$: **opposite** faces, maximizing/minimizing $\boldsymbol{e}_{n+1}$ on P.
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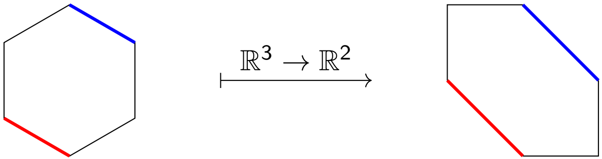

Theorem (Frank '11)

Deformed permutahedra are ***uniquely*** *determined by their pair of top and bottom faces.*
Thus, $P \mapsto (P^+, P^-)$ *is a bijection:*

What's the reciprocal bijection?

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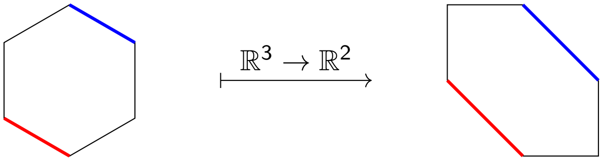

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Thus, $\mathrm{P} \mapsto (\mathrm{P}^+, \mathrm{P}^-)$ *is a bijection:* $\mathbb{SC}_{n+1} \simeq$ *(roughly* $\mathbb{SC}_n^2$*)*

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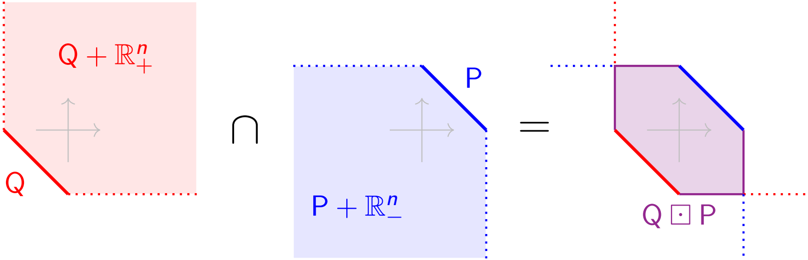
GP-sum: $\mathsf{Q} \boxdot \mathsf{P} := (\mathsf{Q} + \mathbb{R}^+) \cap (\mathsf{P} + \mathbb{R}^-)$



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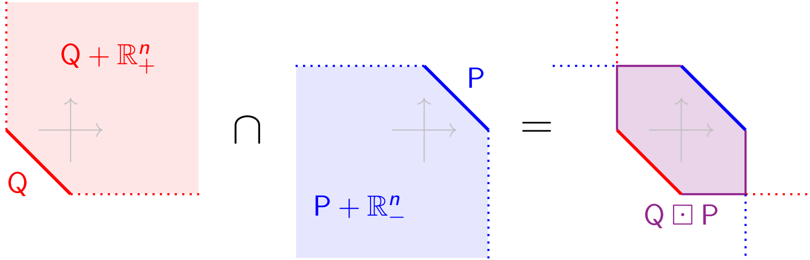
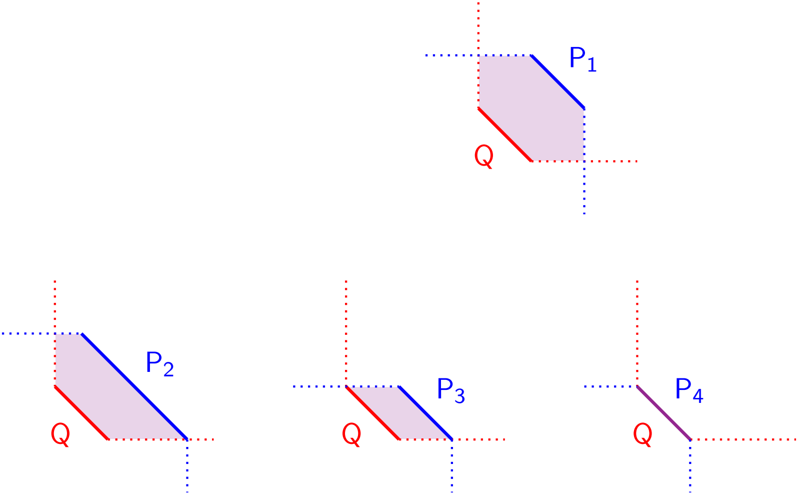


projection of $\boldsymbol{x} \in \mathbb{R}^{n+1}$ is $\downarrow(\boldsymbol{x}) := (x_1, \dots, x_n)$
lift of $\boldsymbol{x} \in \mathbb{R}^n$ is $\uparrow(\boldsymbol{x}) := (x_1, \dots, x_n, -\sum_{i=1}^n x_i)$

Reciprocal of $\mathrm{P} \mapsto (\downarrow(\mathrm{P}^+), \downarrow(\mathrm{P}^-))$ is the map $(\mathrm{P}, \mathrm{Q}) \mapsto \uparrow(\mathrm{Q} \boxdot \mathrm{P})$

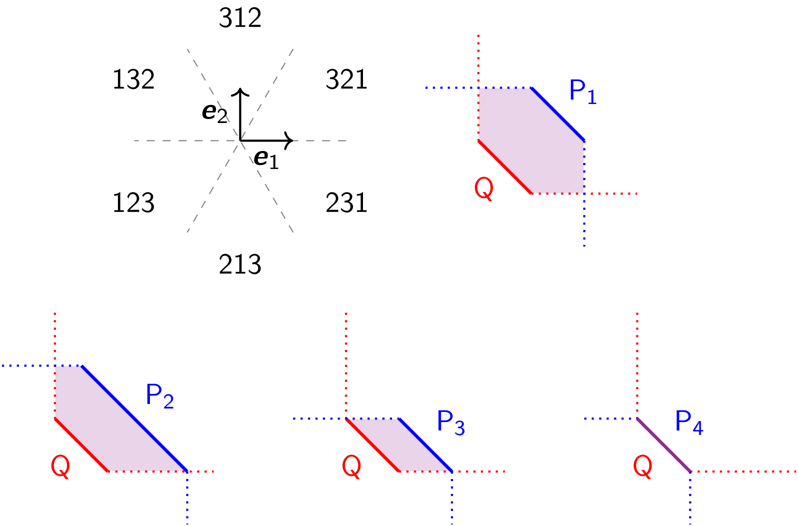
GP-sum exists iff $P \subseteq (Q + \mathbb{R}^+)$ and $Q \subseteq (P + \mathbb{R}^-)$



GP-sums

GP-sum exists iff $\mathrm{P} \subseteq (\mathrm{Q} + \mathbb{R}^{+})$ and $\mathrm{Q} \subseteq (\mathrm{P} + \mathbb{R}^{-})$

iff for all $i \in [n], \sigma \in S_n$, we have $\mathrm{P}_i^{\sigma} \leq \mathrm{Q}_i^{\sigma}$



Theorem (Frank '11, rewritten)

There is a bijection between $\mathbb{SC}_{n+1}$ *and pairs* $(\mathsf{P}, \mathsf{Q}) \in \mathbb{SC}_n^2$ *with* $\forall i, \sigma,\ \mathsf{P}_i^\sigma \leq \mathsf{Q}_i^\sigma$. *Namely:*

$$\begin{aligned} \mathsf{P} &\mapsto \Big({\downarrow}(\mathsf{P}^+), {\downarrow}(\mathsf{P}^-) \Big) \\ (\mathsf{P}, \mathsf{Q}) &\mapsto {\uparrow}(\mathsf{Q} \boxdot \mathsf{P}) \end{aligned}$$

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There is a bijection between $\mathbb{SC}_{n+1}$ *and pairs* $(\mathrm{P}, \mathrm{Q}) \in \mathbb{SC}_n^2$ *with* $\forall i, \sigma,\ \mathrm{P}_i^\sigma \leq \mathrm{Q}_i^\sigma$. *Namely:*

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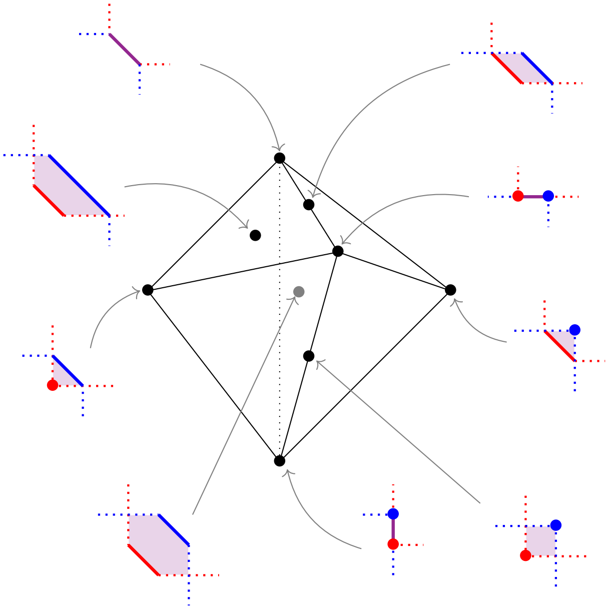
Problem: inductive process on $\mathbb{SC}_n$, not on the face lattice of $\mathbb{SC}_n$

Not inductive process on face lattice of $\mathbb{SC}_n$



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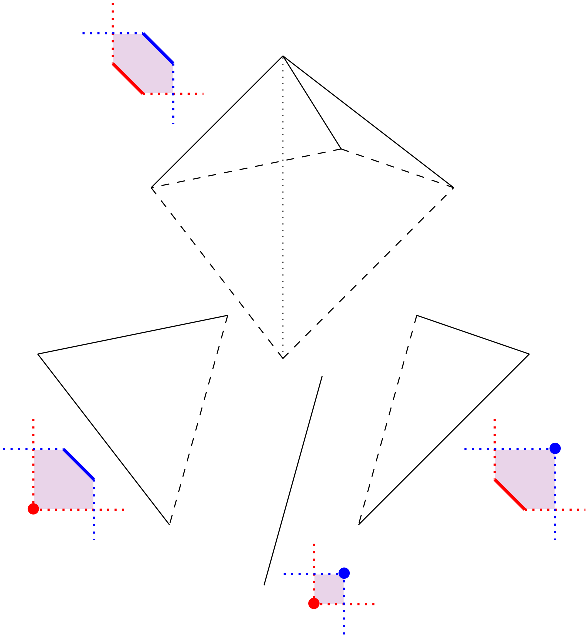

Inductive construction of rays

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P, Q *rays of* $\mathbb{SC}_n$ + *small condition*
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(P, Q) is *fertile* iff there exists $i \in [n]$ such that for all $\tau \in S_n$, either $\mathrm{P}_i^\tau \neq \min_{\sigma \in S_n} \mathrm{P}_i^\sigma$ or $\mathrm{Q}_i^\tau \neq \max_{\sigma \in S_n} \mathrm{Q}_i^\sigma$

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Lemma

If (S, P) *is fertile, then* $\big(\uparrow(\mathrm{S} \boxdot \mathrm{R}), \uparrow(\mathrm{Q} \boxdot \mathrm{P})\big)$ *is also fertile.*

Good news: Fertility is hereditary!

Number of rays

Lemma (Consequently)

If $\{\mathrm{P}_1, \mathrm{P}_2, \dots, \mathrm{P}_r\}$ *are rays of* $\mathbb{SC}_n$ *with* $(\mathrm{P}_i, \mathrm{P}_j)$ *fertile for all* i, j*, then there exist* $\lambda_{i,j} > 0, \boldsymbol{t}_{i,j} \in \mathbb{R}^n$ *such that* $\left\{\uparrow((\lambda_{i,j}\mathrm{P}_j + \boldsymbol{t}_{i,j}) \boxdot \mathrm{P}_i) \ ; \ 1 \leq i, j \leq r\right\}$ *are rays of* $\mathbb{SC}_{n+1}$ *also pairwise fertile.*

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I can construct 1 rays of $\mathbb{SC}_{n+1}$ for each *fertile* pair of rays of $\mathbb{SC}_n$ + fertility is conserved

$\Rightarrow$ I construct $\rho^{2^{n-k}}$ rays of $\mathbb{SC}_n$, with $\rho = \#(\text{fertile family of } \mathbb{SC}_k)$

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Theorem (Loho–Padrol–P., '25+)

For $n \geq 5$, *the number of rays of* $\mathbb{SC}_n$ *is bigger than*

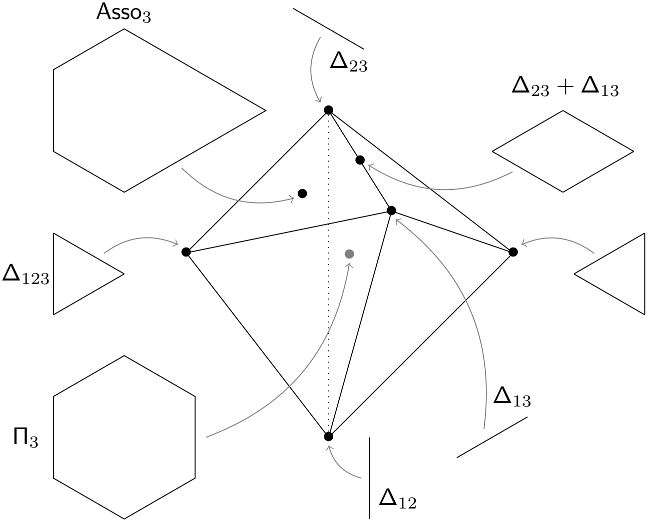
$$656^{2^{n-5}}$$

Going even further beyond

With this induction on (the face lattice of) $\mathbb{SC}_n$:

- Lower bound on the f-vector of $\mathbb{SC}_n$
- Lower bound on the number of non-simplicial faces of $\mathbb{SC}_n$
- New partition of $\mathbb{SC}_n$
- ...

Thank you!



Asso$_3$
Δ_{23}
$\Delta_{23}+\Delta_{13}$
Δ_{123}
Δ_{13}
Π_3
Δ_{12}